Thermoelastic Effects on Some Hollow Structures

Ashwini Kulkarni, R. N. Pakade, Lalsingh Khalsa

***Abstract:** In this paper, an attempt has been made to solve inverse problems of thermoelasticity of a finite length hollow cylinder occupying the space D : α ≤ r ≤ b, -h ≤ z ≤ h. Marchi-Fasulo transform and Hankel transform techniques are used to obtain the general solution for the set of boundary value problems. Particular types of boundary conditions have been taken to illustrate the utility of the approach. The transformed components of the stresses and temperature distribution have been obtained. A numerical inversion technique is employed to invert the integral transform, and the resulting quantities are presented graphically. Key words: Hollow cylinder, Thermoelastic problem, March*

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I. INTRODUCTION

Most materials tend to expand if their temperature rises and, to a first approximation, the expansion and compression is proportional to the temperature change. This temperature changes induced by expansion and compression is based on Thermoelasticity, which is a branch of applied Mathematics, which specially deals with the study of temperature changes and coupling between mechanical deformation and thermal energy calculated in terms of stress. Therefore, a number of theoretical studies concerning them have been reported so far. However, to simplify the analyses, almost all the studies were conducted on the assumption that the upper and lower surfaces of the thin discs or circular are insulated or heat is dissipated with uniform heat transfer coefficients throughout the surfaces. For example, Nowacki, W. [42] has determined steady-state thermal stresses in a thick circular plate subjected to an axis symmetric temperature distribution on the upper face with zero temperature on the lower face and circular edge. Ishihara et al. [23] has considered a circular plate and discussed the transient thermoelastic-plastic bending problem, making use of the strain increment theorem. In all afore mentioned investigations an axis symmetrically heated plate has been considered. Similar studies were also conducted for thick objects. For Example, Nasser, M.EI-Maghraby [39-40] investigated problems due to heat sources in generalized thermoelastic body. Kulkarni, V.S. and Deshmukh K.C. [28] has investigated their research on disc for determining quasi-static thermal stresses in a thick annular disc and circular plates subjected to arbitrary initial temperature on the upper face with lower face atzero.

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II. STATEMENT OF THE PROBLEM

Consider thick circular plate of thickness $2h$ occupying the space $D: 0 \le r \le a,\ -h \le z \le h,$ the material is homogenous and isotropic. The differential equation governing the displacement potential function $\phi(r,z,t)$ as Nowacki [47] is

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=\left(\frac{1+\nu}{1-\nu}\right)\alpha_t T \quad \textbf{(1)}$$

where ν and a_t are Poisson's ratio and linear coefficient of thermal expansion of the material of the plate and T is the temperature of the plate satisfying the differential equation as Noda [41] is

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}+\chi(r,z,t)=\frac{1}{k}\frac{\partial T}{\partial t} \quad \textbf{(2)}$$

Subject to initial condition and boundary conditions

$$T(r,z,t)=F(r,z) \quad \text{at} \quad t=0 \quad \text{for all} \quad 0\le r\le a,\ -h\le z\le h \quad (3)$$

$$\frac{\partial T}{\partial z}=g_1(z,t) \quad \text{at } r=0 \text{ for all } -h\le z\le h,\ t>0 \quad \textbf{(4)}$$

$$\frac{\partial T}{\partial z}=g_2(z,t) \quad \text{at } r=a \text{ for all } -h\le z\le h,\ t>0 \quad \textbf{(5)}$$

$$T+k_1\frac{\partial T}{\partial z}=f_1(r,t) \quad \text{at} \quad z=h \quad \text{for all} \quad 0\le r\le a,\ t>0 \quad \textbf{(6)}$$

$$T+k_2\frac{\partial T}{\partial z}=f_2(r,t) \quad \text{at} \quad z=-h \quad \text{for all} \quad 0\le r\le a,\ t>0 \quad \textbf{(7)}$$

where k is thermal diffusivity of material of the plate.

The displacement function in the cylindrical coordinate system are represented by Love's function as

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 L}{\partial r \partial z} \quad \textbf{(8)}$$

$$u_z = \frac{\partial \phi}{\partial z} + 2(1-\nu)\nabla^2 L - \frac{\partial^2 L}{\partial z^2} \quad \textbf{(9)}$$

The Love's function must satisfy

$$\nabla^2 \nabla^2 L = 0 \quad \textbf{(10)}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad \textbf{(11)}$$

(8)

The component of stresses are represented by the thermoelastic displacement potential (9) ϕ and Love's function L as Noda [41] are

(10)

$$\sigma_{rr} = 2G\left\{\left(\frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left(\nu \nabla^2 L - \frac{\partial^2 L}{\partial r^2}\right)\right\} \quad \textbf{(12)}$$

$$\sigma_{\theta\theta} = 2G\left\{\left(\frac{1}{r}\frac{\partial \phi}{\partial r} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left(\nu \nabla^2 L - \frac{1}{r}\frac{\partial^2 L}{\partial r^2}\right)\right\} \quad \textbf{(13)}$$

$$\sigma_{zz} = 2G\left\{\left(\frac{\partial^2 \phi}{\partial z^2} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left\{\left((z-\nu)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right)\right\}\right\} \quad \textbf{(14)}$$

$$\sigma_{rz} = 2G\left\{\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r}\left\{\left((1-\nu)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right)\right\}\right\} \quad \textbf{(15)}$$

For traction free surface stress function

$$\sigma_{\theta z} = \sigma_{r\theta} = 0 \text{ at } z = \pm h \text{ for thick plate.} \quad (16)$$

Equations to constitute the mathematical formulation of the problem under consideration.

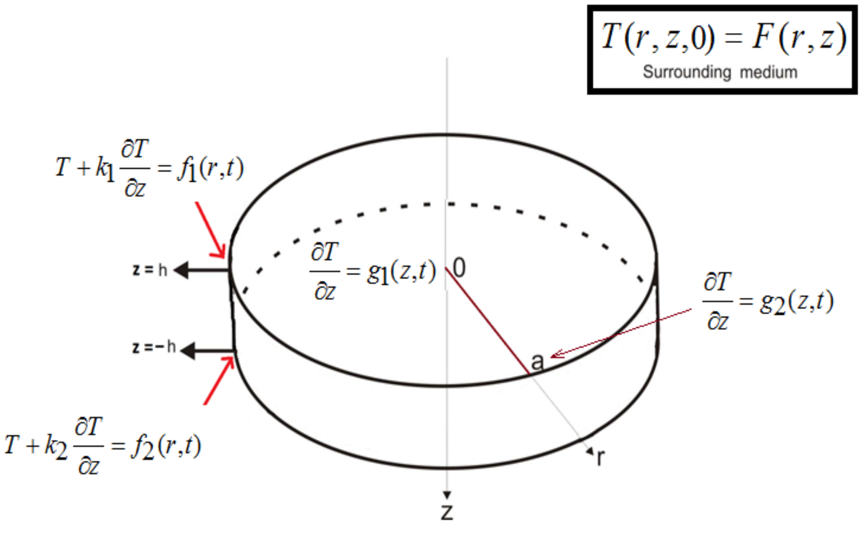

Figure 1: Shows the Geometry of the Problem

III. SOLUTION OF THE PROBLEM

Applying Hankel transform defined to the equation, we get

$$-\xi_m^2 T^*(\xi_m, z, t) + \frac{d^2 T^*}{dz^2}(\xi_m, z, t) + \chi^*(\xi_m, z, t) = \frac{1}{k}\frac{dT^*}{dt} \tag{17}$$

Again applying Marchi-Fasulo transform defined in to above equation, we obtain

$$\frac{d\overline{T}^*}{dt} + kP^2\overline{T}^* = \Psi \tag{18}$$

where

$$P^2 = \xi_m^2 + a_n^2 \tag{19}$$

Equation is a linear equation whose solution is given by

$$\overline{T}^*(\xi_m, n, t) = e^{-kP^2 t}\int_0^t \Psi e^{kP^2 t'} dt^1 + Ce^{-kP^2 t} \tag{20}$$

Using (3), we get

$$C = F^*(m, n) \tag{21}$$

Thus we have

$$\overline{T}^*(\xi_m, n, t) = e^{-kP^2 t}\left[\int_0^t \Psi e^{kP^2 t^1} dt^1 + \overline{F}^*(m, n)\right]$$

Applying inversion of Marchi-Fasulo transform and Hankel transform to the differential equation, we get

$$T(r, z, t) = \frac{2}{a^2}\sum_m\sum_n \frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n(z)}{\lambda_n}e^{-kP^2 t}$$

$$\times\left[\int_0^t \Psi\, e^{kP^2 t'} dt' + \overline{F}^*(m, n)\right] \tag{22}$$

This is the desired solution of the given problem. Let us assume Love's function L, which satisfy condition as

$$L(r, z) = \frac{2}{a^2}\sum_m\sum_n \frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n(z)}{\lambda_n}\Omega \tag{23}$$

where

$$\Omega = e^{-kP^2 t}\left[\int_0^t \Psi e^{kP^2 t'} dt' + \overline{F}^*(m, n)\right] \tag{24}$$

we get displacement potential ϕ *as*

$$\phi = A\sum_m\sum_n \frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\left[\frac{P_n'(z)}{\lambda_n}\Omega + B(t)\right] \tag{25}$$

where

$$A=\left(\frac{1+\nu}{1-\nu}\right)\frac{2\alpha_t}{a^2}\ , \qquad (26)$$

$$B(t)=\int e^{-kP^2 t}\left(\int_0^t \Psi\ e^{-kP^2 t'}dt'+\overline{F}^*(m,n)\right)dt \qquad (27)$$

IV. DETERMINATION OF DISPLACEMENT FUNCTION

Substituting, we get

$$u_r = A\sum_m\sum_n \frac{\xi_m J_1(r\xi_m)}{[J_1(a\xi_m)]^2}\left[\frac{P_n'\ (z)}{\lambda_n}\psi+B(t)\right]$$

$$-\frac{2}{a^2}\sum_m\sum_n \frac{\xi_m J_1(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n'\ (z)}{\lambda_n}\Omega$$

and

$$u_z = A\sum_m\sum_n \frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\left[\frac{P_n''\ (z)}{\lambda_n}\Omega+B(t)\right]$$

$$+2(1-\nu)\left[\frac{2}{a^2}\sum_m\sum_n \frac{\xi_m^2[J_1'\ (r\xi_m)+J_1(r\xi_m)]}{[J_1(a\xi_m)]^2}\frac{P_n(z)}{\lambda_n}\Omega\ \right]$$

$$+\frac{2(1-2\nu)}{a^2}\sum\sum \frac{J_0(r\xi_m)}{J_1(a\xi_m)^2}\frac{P_n''\ (z)}{\lambda_n}\Omega \qquad (24)$$

Substituting in above equations, we obtain

$$\sigma_{rr}=2G\left\{\frac{2(\nu-1)}{a^2}\sum_m\sum_n \frac{\xi_m^2 J_1'\ (r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n'\ (z)}{\lambda_n}\Omega\right.$$

$$+\frac{2}{a^2 r}\sum_m\sum_n \frac{\xi_m J_1(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n'\ (z)}{\lambda_n}\Omega$$

$$+\frac{2}{a^2}\sum_m\sum_n \frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n'\ (z)}{\lambda_n}\Omega$$

$$-\frac{A}{r}\sum_m\sum_n \frac{\xi_m J_1(r\xi_m)}{J_1(a\xi_m)^2}\left[\frac{P_n'\ (z)}{\lambda_n}\Omega+B(t)\right]$$

$$\left.-A\sum_m\sum_n \frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\left[\frac{P_n'\ (z)}{\lambda_n}\Omega+B(t)\right]\right\}$$

$$\sigma_{\theta\theta}=2G\left\{\frac{2\nu}{a^2}\sum_m\sum_n \frac{\xi_m^2 J_1^1(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n^1(z)}{\lambda_n}\Omega\right.$$

$$+\frac{2(\nu-1)}{a^2}\sum\sum \frac{\xi_m J_1(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n^1(z)}{\lambda_n}A\Omega$$

$$+\frac{2\nu}{a^2}\sum_m\sum_n\frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n^{111}(z)}{\lambda_n}\Omega$$

$$-A\sum_m\sum_n\frac{J_0(r\xi_m)}{J_1(a\xi_m)^2}\left[\frac{P_n^1(z)}{\lambda_n}\Omega+B(t)\right]$$

$$-A\sum_m\sum_n\xi_m^2\frac{J_1^1(r\xi_m)}{[J_1(a\xi_m)]^2}\left[\frac{P_n^1(z)}{\lambda_n}\Omega+B(t)\right]\Bigg\}$$

$$\sigma_{zz}=2G\Bigg\{\frac{(2-\nu)}{a^2}\sum\sum\frac{\xi_m^2J_1^1(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n^1(z)}{\lambda_n}\Omega$$

$$+\frac{2(2-\nu)}{a^2r}\sum_m\sum_n\frac{\xi_mJ_1(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n^1(z)}{\lambda_n}\Omega$$

$$+\frac{2(1-\nu)}{a^2}\sum\sum\frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n^{111}(z)}{\lambda_n}\Omega$$

$$-A\sum\sum\frac{\xi_m^2J_1^1(r\xi_m)}{[J_1(a\xi_m)]^2}\left[\frac{P_n^1(z)}{\lambda_n}\Omega+B(t)\right]$$

$$-\frac{A}{r}\sum\sum\frac{\xi_mJ_1(r\xi_m)}{[J_1(a\xi_m)]^2}\left[\frac{P_n^1(z)}{\lambda_n}\Omega+B(t)\right]\Bigg\}$$

$$\sigma_{rz}=2G\Bigg\{\frac{2(1-\nu)}{a^2}\sum\sum\frac{\xi_m^3J_1^{11}(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P^1(z)}{\lambda_n}\Omega$$

$$+\frac{2(-\nu)}{a^2}\sum\sum\frac{\xi_mJ_1(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n^{11}(z)}{\lambda_n}\Omega$$

$$+\frac{2(1-\nu)}{a^2}\sum\sum\frac{\xi_m}{[J_1(a\xi_m)]^2}\left\{\frac{\xi_mrJ_1^1(r\xi_m)}{r_2}\right\}\frac{P_n(z)}{\lambda_n}\Omega$$

$$+A\sum\sum\frac{\xi_mJ_1(r\xi_m)}{[J_1(a\xi_m)]^2}\left[\frac{P_n^4(z)}{\lambda_n}\Omega+B(t)\right]\Bigg\}$$

(28)

where

$$A=\left(\frac{1+\nu}{1-\nu}\right)\frac{2\alpha_t}{a^2},$$

$$\Omega=e^{-kp^2t}\left[\int_0^t\Psi e^{-kp^2t^1}dt^1+\overline{F}^*(m,n)\right],$$

$$B(t)=\int\Omega\,dt$$

V. SPECIAL CASE

Set $F(r,z)=z^2(1-r^2)$

Applying Marchi-Fasulo transform, are obtain

$$\overline{F}(r,n)=(1-r^2)\int_{-h}^{h}z^2P_n(z)dz$$

$$\overline{F}(r,n)=(1-r^2)\Phi_n\left[\frac{2h^2\sin(a_nh)}{a_n}+\frac{4h\cos(a_nh)}{a_n^{\,2}}-\frac{4\sin(a_nh)}{a_n^{\,3}}\right] \quad (30)$$

where

$$P_n(z)=Q_n\cos(a_nz)-W_n\sin(a_nz),$$

$$Q_n=a_n(\alpha_1+\alpha_2)\cos(a_nh)+(\beta_1-\beta_2)\sin(a_nh)$$

$$W_n=(\beta_1-\beta_2)\cos(a_nh)+a_n(\alpha_1-\alpha_2)\sin(a_nh)$$

Again on applying Hankel transform, we obtain

$$\overline{F}^*(m,n)=\Pi_n\left[\frac{a}{\xi_m}J_1(a\xi_m)-\frac{a(a^2\xi_m^2-4)}{\xi_m^3}J_1(a\xi_m)-\frac{2a^2}{\xi_m^2}J_0(a\xi_m)\right]$$

(31)

where

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$$\Pi_n = \Phi_n\left[\frac{2h^2\sin(a_n h)}{a_n} + \frac{4h\cos(a_n h)}{a_n^{\ 2}} - \frac{4\sin(a_n h)}{a_n^{\ 3}}\right]$$

and

$$\Phi_n = a_n(\alpha_1+\alpha_2)\cos(a_n h) + (\beta_1-\beta_2)\sin(a_n h).$$

Using equation, one obtains

$$T(r,z,t) = \frac{2}{a^2}\sum_m\sum_n \frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n(z)}{\lambda_n}e^{-kp^2t} \times\left[\int_0^t \Psi e^{kp^2t}\,dt^1 + \Pi_n\right]$$

$$\times\left(\frac{a}{\xi_m}J_1(a\xi_m) - \frac{a(a^2\xi_m^2-4)}{\xi_m^{\ 3}}J_1(a\xi_m) - \frac{2a^2}{\xi_m^2}J_0(a\xi_m)\right)$$

VI. NUMERICAL RESULTS

Set $a = 0\cdot 2m, k = 15.9\times10^{-6}, t = 1$ we get

$$T(r,z,t) = \frac{1}{2}\sum_m\sum_n \frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n(z)}{\lambda_n}e^{-(15.9\times10^6)P^2t}$$

$$\times\left[\int_0^1 \Psi e^{(15.9\times10^6)P^2t^1}\,dt^1 + \Pi_n\left(\frac{2}{\xi_m}J_1(2\xi_m) - \frac{2(4\xi_m^2-4)}{\xi_m^{\ 3}}J_1(2\xi_m) - \frac{2}{\xi_m^2}J_0(2\xi_m)\right)\right]$$

As an illustration, we carried out numerical calculations for a thick circular plate made up of aluminum metal (refer Table 1 for parameter) and examine the thermoelastic behavior in the state for temperature distribution, displacement and thermal stresses in radial and axial direction.

Table 1: Thermal Material Properties

Materials	Symbol	K	Cp	ρ	α	λ	E	ν
		Btu/ hr ft^0 F	Btu/ lb ^{0} F	lb/ ft^3	ft$^{2/}$ hr	1/F (X 10^{-6})	GP$_a$	
Aluminum	Al	117	0.208	169	3.33	12.84	70	0.35
Copper	Cu	224	0.091	558	4.42	9.3	117	0.36
Iron	Fe	36	0.104	491	0.70	6.7	193	0.21
Silver	Ag	242	0.056	655	6.60	10.7	83	0.37

Figure 2 shows that the variation of temperature distribution along axial direction, it is clear that temperature decreases initially at time t = 0.25, 0.50, 0.75, 1.00 and slightly increasing at z = 2.5, the curve behaves like a sinusoidal type. But due to the ax symmetric internal heating at t = 0.5 temperature decreases upto zero at z = 3.

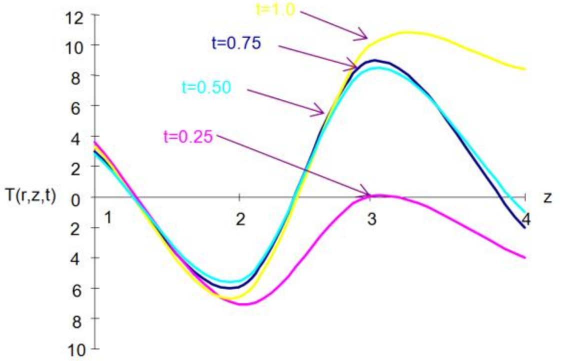

Figure 2. Temperature Distribution Along Axial Direction

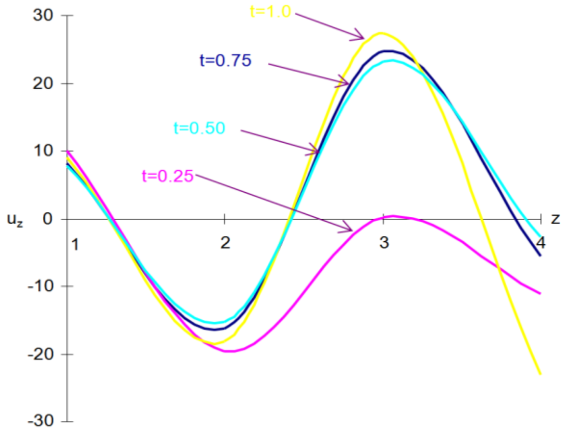

Figure 3. Displacement Along Axial Direction

In figure 3 depicts the variation of displacement u_z along axial direction, it is clear that radial displacement decreases initially at time t = 0.25, 0.50, 0.75, 1.00 and slightly increasing at z = 2.5 and attain peak value for z = 3, again the curve behaves like a sinusoidal type. But due to the axis symmetric internal heating at t = 0.5 displacement decreases upto zero at z = 3.

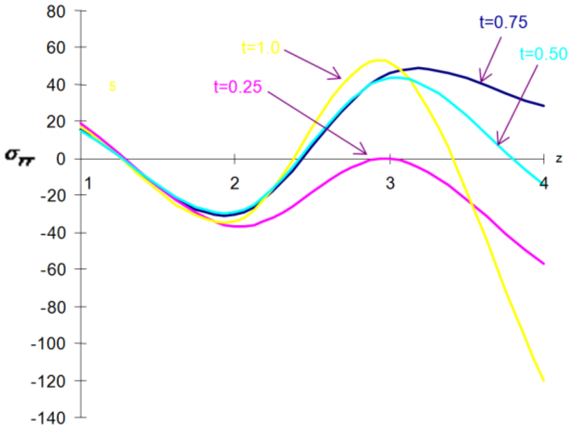

Figure 4. Radial Stresses Along Axial Direction

In figure 4 displays the variation of radial stresses along axial direction at different values of time, it is clear that radial stresses initially decreases at time t = 0.25, 0.50, 0.75, 1.00 and the start increasing at z = 2.5 and attain peak value for z = 3, again the curve behaves like a sinusoidal type. But due to the axis symmetric internal heating at t = 0.5 temperature decreases upto zero at z = 3. It is also observed that the Axial stresses σ_{zz} and the Shear Stresses σ_{rz} along axial direction for different values of time were found similar to that of Radial stresses σ_{rr} and Tangential stresses σ□□ along axial direction with only slight change in the magnitude.

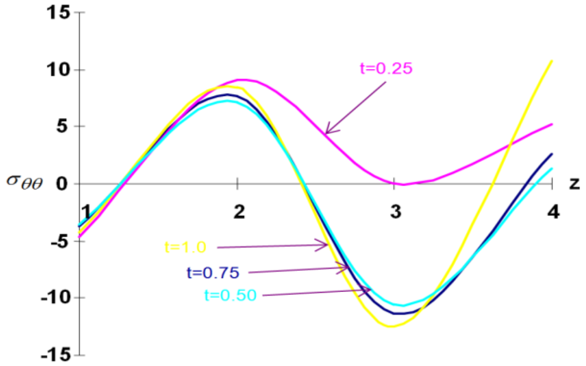

Figure 5. Tangential stresses along axial direction

VII. CONCLUSION

The temperature distribution, displacement and thermal stresses of thick circular plate are investigated with known boundary conditions. Finite integral transform techniques are used to obtain numerical results. Any particular cases of special interest can be assigned to the parameters and functions in expressions. The temperature, displacement and thermal stresses that are obtained can be useful to the design of structure or machines in engineering applications.

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