Lifting Scheme Based Designing of Wavelets in Spiral Addressing Model on a Hexagonal Grid

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Abstract- Image processing in hexagonal grid is very much advantageous than in the conventional rectangular grid. The advantages include higher angular resolution, consistent connectivity and higher sampling efficiency. A wide class of operations on images can be performed directly in the wavelet domain by operating on its coefficients of the images. Operating in wavelet domain enables to operate on different resolutions, manipulate features at different scales and localize the operation in both spatial and frequency domains. A new method of designing hexagonal wavelets using lifting scheme in the spiral addressing scheme is proposed in this thesis. It is computationally efficient because they are not based on Fourier transforms, and could be performed in place.

Index Terms:- *wavelets,lifting scheme,spiral addressing, hexagonal grid*

I. INTRODUCTION

Image processing uses a rectangular grid for image representation and processing. Hexagonal grid is an alternative pixel tessellation scheme besides the conventional square grid for sampling and representing images. Using hexagonal grids to represent digital images have been studied for more than 40 years. Increased processing capabilities of graphic devices and recent improvements in CCD technology have made hexagonal sampling attractive for practical applications. Continuous studies in this field brought new interests to this topic. A hexagonal coordinate system is simply a system which replaces the common square lattice and describes the images in favor of a hexagonal lattice. From the perspective of computer vision, hexagonal coordinate system closely resembles the layout of photo-receptors in the human retina. Research suggests that the simulation of at least some of the capabilities possessed by human eye and the visual processing areas of the brain can be easily executed on the images that are laid out on a hexagonal lattice. Sampling on a hexagonal lattice is a promising solution which has been proved to have better efficiency and less aliasing [1]. Its computational power for intelligent vision pushes forward the image processing field. Many reports describing the advantages of using such a grid type are found in the literature. The major advantages are higher degree of circular symmetry, uniform connectivity, greater angular resolution, lesser storage and reduction in computation for image processing operations. Mersereau [2] has shown that for circularly band limited signals, 13.4% fewer sampling points are required with the hexagonal grid to maintain the equal information with the rectangular grid.

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Many resampling techniques were proposed like brick wall, quincunx sampling, least squares approximation of splines, etc [3]. Using wavelets an image can be decomposed into a multiresolution hierarchy of localized information at different spatial frequencies. Operating in the wavelet domain enables one to perform various operations progressively in a coarse-to-fine fashion, operate on different resolutions, manipulate features at different scales, trade off accuracy for speed, and localize the operation in both the spatial and the frequency domains. Lifting scheme [5],[6],[7] is a simple and efficient technique for the creation of wavelets. It is computationally efficient because they are not based on Fourier transforms and could be calculated in-place. In this work, the first part deals with basics of hexagonal image processing followed by the spiral addressing scheme. Then it covers the construction of wavelets using lifting scheme. We propose one new method of designing hexagonal wavelets using lifting scheme in the spiral addressing scheme.

II. HEXAGONAL SAMPLING SCHEME

A digital image *a* [*m*, *n*] described in a 2-D discrete space is derived from an analog image *a* (*x*, *y*) in a 2D continuous space through a *sampling* process that is frequently referred to as digitization. The 2D continuous image *a* (*x*, *y*) is divided into *N rows* and *M columns*. The intersection of a row and a column is termed a *pixel*. The value assigned to the integer coordinates [*m*, *n*] with { *m*=0,1,2,…,*M*–1 } and { *n*=0,1,2,…,*N*–1} is *a* [*m*, *n*]. Following are the various tessellation schemes used for the digitization of the image.

2.1 Three possible regular tessellation schemes

There exist only three possible regular tessellation schemes to tile a plane without overlapping among the samples and gaps between them, namely the tessellation with hexagons, with squares, and with regular triangles (Fig.1). Any other types of spatial tessellation will result in either unequal distance between neighboring pixels, or introduce gaps or overlaps among samples.

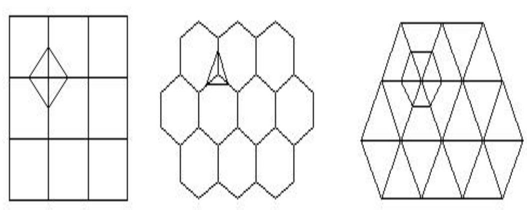

Fig. 1 Three schemes of regular tessellation

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2.2 Efficient Sampling Scheme

An insufficient sampling rate can always introduce unwanted effects in the reconstructed signal, referred as aliasing. Middleton [8] investigated sampling and reconstructing wave number limited multi-dimensional functions of signals. They concluded that the rectangular lattice is not the most efficient sampling lattice which uses minimum number of sampling points to achieve exact reconstruction.A similar conclusion was obtained by Mersereau [2] and Vitulli [5], who showed that for signals which are band-limited over a circular region in Fourier space, 13.4% fewer sampling points are required with the hexagonal grid to maintain equal high frequency image information with the rectangular grid, thus less storage and less computation time are required. An example is that in image coding application, one may expect that the coding efficiency can be increased by using the hexagonal sampling scheme. On the hexagonal grid, digitization displays a better connectivity and is perceived as being approximated by small poly lines, whereas on the square grid, digitization is still perceived as being approximated by pixels. Such a perception of single pixels disturbs the impression of continuity of the discretized line. This is due to the fact that in the square grid neighbors of a pixel are not placed all at the same distance. Moreover, two diagonal neighbors in the square grid have only one point in common, whereas two horizontal or vertical neighbors of the square grid, and all the neighbors of a pixel in the hexagonal grid, have one segment in common with their neighbor.

III. SPIRAL ADDRESSING SYSTEM

Middleton and Sivaswamy [8] proposed a one-dimensional addressing system, as well as two operations based on this addressing system, for hexagonal structure. This system is called as Spiral Architecture (Fig.2). Spiral Architecture (SA) is inspired from anatomical consideration of the primate's vision system.

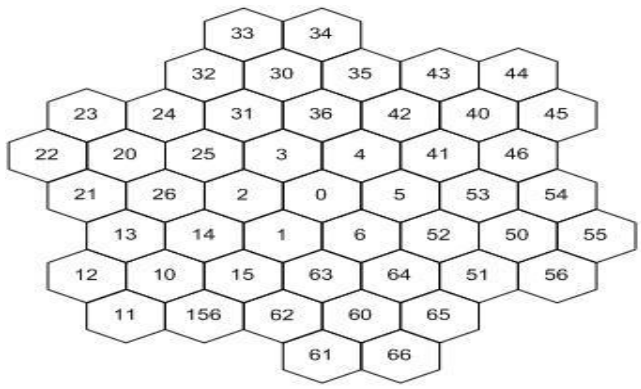
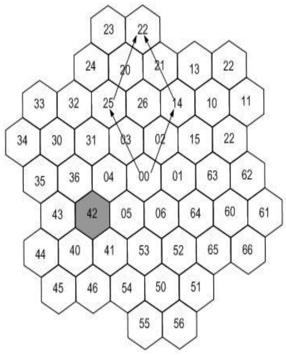


Fig. 2 Spiral addressing

The address in the spiral architecture grows from the centre of image in powers of seven along a spiral like curve. This addressing scheme combined with two mathematical operations, spiral addition and spiral multiplication is the base of Spiral Architecture. The spiral addition and spiral multiplication correspond to image translation and image rotation respectively. Middleton and Sivaswamy [8] also proposed a single-index system for pixel addressing by modifying the Generalized Balanced Ternary system, as shown in Fig.3.



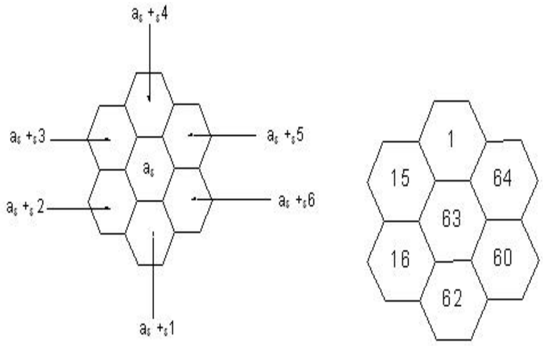
25
+14
22

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	63	15	2	0	6	64
2	2	15	14	26	3	0	1
3	3	2	26	25	31	4	0
4	4	0	3	31	36	42	5
5	5	6	0	4	42	41	53
6	6	64	1	0	5	53	52

(a) (b)

Fig. 3 (a) Hexagonal image structure with indices (b) Balanced ternary addition

Neighborhood operations are often used in image processing. Finding the neighbor in a hexagonal image makes use of the spiral addition operation [8]. In a seven-pixel cluster, the neighborhood relation can be determined by spiral addition as follows.



(a) Neighborhood relationship (b) An example of neighborhood

Fig. 4 Neighborhood relationship with spiral architecture

Let the spiral address of the central pixel, as shown in Fig. 4(a), be denoted by a, Then the spiral address of its neighbor pixel can be described by spiral addition denoted by +, with a certain number of displacements, as shown in Fig. 4 (a). An example is given in Fig. 4(b). For the whole image, the spiral rotation direction is as shown in Fig. 5, one can find out the spiral address of any hexagonal pixel with centre on a certain hexagonal pixel whose spiral address is known.

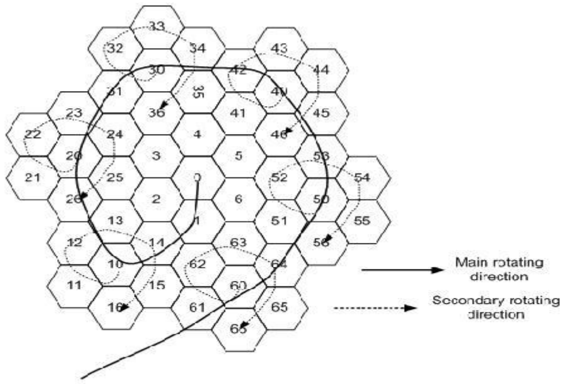


Fig. 5 Spiral rotation direction

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The Spiral Architecture has some distinguishing features compared to the square image processing. The one dimensional addressing scheme leads to an efficient storage and the placement of the origin at the centre of the image simplifies geometric transformations of a given image. Hexagonally sampled image allows non-traditional neighborhoods with consistent boundary connectivity, which is useful for many computer vision applications.

IV. WAVELETS

Wavelets were developed independently in the fields of mathematics, quantum physics, electrical engineering and seismic geology. Interchanges between these fields during the last ten years have led to many new wavelet applications such as image compression, human vision, radar, earth quake prediction and various other image processing operations. The transform is computed at various locations of the signal and for various scales of the wavelet. If the process is done in a smooth and continuous fashion (i.e., if scale and position is varied very smoothly) then the transform is called Continuous Wavelet Transform (CWT). If the scale and position are changed in discrete steps, the transform is called Discrete Wavelet Transform (DWT). Mathematically, a wavelet can be denoted as

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right) \qquad (1)$$

b – Location parameter

a – scaling parameter

For the function to be a wavelet, it should be time limited. For a given scaling parameter 'a', we translate the wavelet by varying the parameter 'b'. Design of wavelets is done in two methods. One is the filter bank method which uses the filters, analysis and synthesis filter banks and the other is the lifting scheme method. Here we consider the lifting method and how it is used to construct wavelets in spiral addressing scheme. The lifting scheme [5] is a method for constructing wavelets in the spatial domain. It consists of three steps:

1) Splitting the data into two subsets,

2) Computing the wavelet coefficients as the failure to predict one subset based on the other (high pass),

3) Computing the scaling function coefficients by updating the remaining subset (low pass).

Any discrete wavelet transform can be factored into lifting steps [9], thus allowing in-place computation of the wavelet transforms, faster computation, asymptotically reducing the complexity by a factor of four and construction of wavelet transforms that map integers to integers [10].

V. CONSTRUCTION OF WAVELETS USING LIFTING METHOD

We present the construction of wavelets on a hexagonal lattice based on the method of lifting which is an efficient way to compute wavelets[11]. In contrast to Laine [12] the hexagonal wavelet thus derived does not use a Fourier transform method. Lifting scheme here explained is based on the spiral addressing scheme which is implemented on the HIP framework explained in [8]. Construction of the wavelet is performed in three phases: splitting, predicting, and updating. Splitting partitions the data into two subsets A and B. Predicting computes wavelet coefficients at A using the points in B. Updating changes the points in B in order to preserve the mean value. All, these operations are computed in-place and reversing them can produce the inverse transform. A complete wavelet transform of a hexagonal image requires repeated application of the splitting, predicting, and updating steps. These are now described via the first case for a two layer HIP image $h(x)$, $x \in G_2$ and a general case.

5.1 First case

Initially, the lifting scheme partitions the data into even and odd pairs corresponding to up sampling and down sampling the data by a factor of 2. Specifically for an image on a square lattice this corresponds to a partitioning scheme based upon the 4-neighbours of a point. However, due to the topology of the hexagonal lattice this is not really plausible. In fact the 3 symmetric axes make it impossible to provide such a partitioning using any of the 6-neighbours of a hexagonal point. Thus, the method illustrated in Fig. 6(a) is performed.

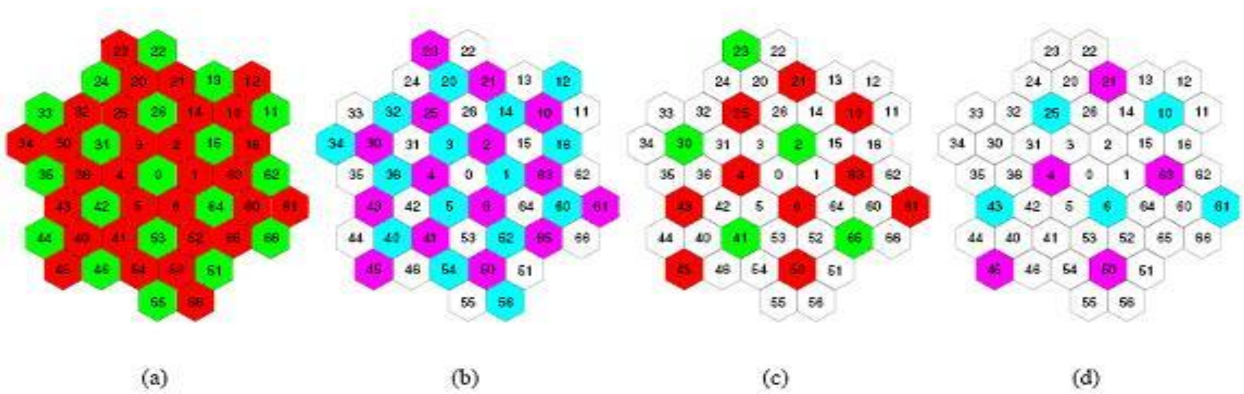

Fig. 6 Partitioning of the hexagonal image using lifting

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The green points ({0, 15… 66}), collectively denoted as A_0, are analogous to up sampling, while the red points ({1, 2….. 65}), denoted as B_0, are analogous to down sampling. Using the following set operation :

$$S(x)=\{x, x+15, x+26, x+31, x+42, x+53, x+64\} \tag{2}$$

Here '+', is the addition operation defined in the context of the HIP framework. The set of points in A_0 can thus be defined recursively as follows:

$$A_0 = S_n(0) = S_{n-1}(0) \bigcup S_{n-1}(15) \bigcup \cdots\cdots S_{n-1}(64) \tag{3}$$

B_0 can be defined simply as $G^0 - A_0$. The next step in lifting is to predict the values at addresses given in A_0 using the nearest neighbors in B_0. This can be performed using a linear interpolation. Thus for a given member in A_0 labeled a_i the prediction step is :

$$h(a_i) = h(a_i) - \frac{1}{6}\sum_{g\in N_a^B} h(g) \tag{4}$$

Here, g are the nearest neighbors in B_0 to point $a_i \varepsilon A_0$. In this case it corresponds to $N_a^B = \{a_i+1, \cdots, a_i+6\}$

The values for $h(a_i)$ are omputed in-place. Once this has been computed the values in B_0 can be updated to preserve the image mean, via

$$h(b_i) = h(b_i) + \frac{1}{9}\sum_{g\in N_b^A} h(g) \tag{5}$$

Here, g is the nearest neighbors in A_0 to point $b_i \varepsilon B_0$. In this case it corresponds to $\{b_i+1, b_i+3, b_i+5\}$ *or* $\{b_i+2, b_i+4, b_i+6\}$ depending on the value of b_i.

In the normal lifting scheme the steps just described would be reapplied repeatedly to the remaining low frequency parts of the image (B_i). However, examination of Fig.6(a) shows that the remaining points do not have the same topology as the original. In fact, it looks like a group of hexagonal rings. For this reason we have to reexamine the splitting, predicting, and updating steps. As each hexagonal ring consists of 6 points it is possible to just split the space in half but again due to topological constraints it is not possible to choose odd and even points. The specific partition is illustrated in Fig. 6(b). The cyan points (1, 3,… 60), denoted by C_0 are the high frequency points and the magenta points (2, 4,…,65), denoted by D_0 are the low frequency points. As the spacing between points in C_0 is equivalent to that in A_0 the points can be recursively defined:

$$C_0 = S_n(1) = S_{n-1}(1) \bigcup S_{n-1}(16) \bigcup \cdots\cdots \bigcup S_{n-1}(63) \tag{6}$$

We can notice the change of origin. The set of addresses corresponding to the low frequency image data is defined as $D_0 = B_0 - C_0$. The predicted values of c_i can be computed as a linear interpolation, thus:

$$h(c_i) = h(c_i) - \frac{1}{3}\sum_{g\in N_c^D} h(g) \tag{7}$$

Here, g are the nearest neighbors in D_0 to point $c_i \varepsilon C_0$. In this case it is $\{c_i+1, c_i+3, c_i+5\}$. Similarly we can compute the updating step as follows:

$$h(d_i) = h(d_i) + \frac{1}{6}\sum_{g\in N_d^C} h(g) \tag{8}$$

Here, g are the nearest neighbors in C_0 to point $d_i \varepsilon D_0$. In this case it is $\{d_i+2, d_i+4, d_i+6\}$.

Notice the similarity between the two sets N_c^D and N_d^C and the one employed in the update step for B_0. This implies that the update step for B_0 thus requires just prior knowledge of the neighbors for C_0 and D_0 to compute.

5.2 General case

At the end of the first iteration of the lifting procedure we have computed the high frequency values corresponding to A0 and C0 are left with set D0 from which to continue the process. There are many possibilities but the only requirement is that it should be simple to apply. Examinations of Fig. 6(c), which are the points in D0, show two features. Firstly, the original origin is no longer in the data set so a shift is required. Secondly, the radius of the hexagonal ring has expanded by √3 and rotated by 300. Fortunately, due to the vector nature of HIP addresses the shift can just be accommodated by an addition of the address to the origin and the rotation and scaling can be accommodated by a multiplication.

There are many valid possibilities that can be used but for the work presented here an offset of 2 and a scaling of 15 were chosen.

Using these ideas the general case can be derived. This is achieved by rewriting the cases for the first case and introducing an origin shift, 'o', and a transformation, 'r', defined as:

$$o_0 = 0, \quad o_n = 15 o_{n-1} + 2$$

$$r_n = \prod_{i=0}^{n} 15 \qquad (9)$$

Using these, steps previously outlined can be redefined. By redefining the partitioning operation to be

$$S^m(y) = \{y, y+15r_m, \cdots\cdots, y+64r_m\} \qquad (10)$$

where $y = x + o_m$.

The first splitting is:

$$A_m = S_n^m(o_m)$$

$$B_m = \begin{cases} G^\lambda - A_m, & m = 0 \\ D_{m-1} - A_m, & else \end{cases} \qquad (11)$$

The second splitting is :

$$C_m = S_n^m(o_m + 1)$$
$$D_m = B_m - C_m \qquad (12)$$

The neighborhoods are also changed to include these factors as follows:

$$N_a^B = \{a_i + 1r_m, \cdots, a_i + 6r_m\}$$

$$N_b^A = \begin{cases} \{b_i + 1r_m, b_i + 3r_m, b_i + 5r_m\}, & b_i \in D_m \\ \{b_i + 2r_m, b_i + 4r_m, b_i + 6r_m\}, & b_i \in C_m \end{cases}$$

$$N_c^D = \{b_i + 1r_m, b_i + 3r_m, b_i + 5r_m\}$$

$$N_d^C = \{b_i + 2r_m, b_i + 4r_m, b_i + 6r_m\} \qquad (13)$$

Equations given in the prediction and updating section can now be applied directly to compute the wavelet. The inverse wavelet uses the same sets of addresses though the signs are swapped in the prediction and updating steps.

VI. IMPLEMENTATION

In this approach we use the Hexagonal Image Processing (HIP) framework which is implemented in Python language. Using this, image can be resampled into hexagonal using the spiral addressing scheme. Construction of wavelets using the lifting scheme is done in this framework. In this framework number of layers required to represent an image with M x N size is approximately $\frac{\log(M)+\log(N)}{\log 7}$.

(a) original image **(b) 2-layer** **(c) 3-layer** **(d) 4-layer** **(e) 5-layer**

Fig 7 Hexagonal sampling of flower. jpg image in HIP framework

Original size of the image is 128 x 128. So the number of layers required to display this image fully in HIP framework is given by

$$\lambda = \frac{\log 128 + \log 128}{\log 7} = \frac{4.214}{0.845} = 4.987 \approx 5.$$

VII. CONCLUSION

The possibility of constructing wavelets on spiral addressing scheme using lifting method was studied extensively. Since there is no dedicated hardware available for hexagonal-based image capture and display, conversion has to be done from square to hexagonal image before hexagonal-based image processing. The difference will be clear only if we have hexagonal based image capture and display systems.

The use of these wavelets can be extended to many pattern recognition operations like object recognition and segmentation. The properties of the hexagonal wavelets are to be studied extensively in order to apply it to multiresolution image processing operations.

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