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Generalized Rotation Tensor of An Arbitrary Spatial System of Forces

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***Abstract:** The proposed article presents an extension of the well-known theorem of theoretical mechanics about three moments, which is valid for an arbitrary planar system of forces, to the general case of an arbitrary spatial system of forces. Existence and uniqueness theorems for a symmetric static tensor of moments are formulated with a presentation of their proofs. For an arbitrary spatial system of forces, the dynamic tensor of moments is also formulated. A technique is presented for determining the principal directions and principal values of the moment tensor, for which the number of its independent components is reduced to three. This case provides clear evidence for the existence of a generalized rotation. A concrete example of an arbitrary system of forces is given, confirming the equivalence of the conditions of static equilibrium in the classical and new interpretations.*

***Keywords:** Second Form of Equilibrium, Arbitrary Spatial System of Forces, Moment Tensor, Generalized Rotation, Generalized Rotation Tensor.*

Abbreviations:
M p: Primary Moment

I. INTRODUCTION

It is known [1] that the equilibrium condition for an arbitrary spatial system of forces is formulated by the fundamental theorem of statics [2], or the Poinsot theorem, according to which the primary vector and the primary moment of forces acting on the rigid body concerning some center of reduction P (Fig. 1), i.e [3].

$$\begin{cases} \vec{R}' = 0; \\ \vec{M}_P = 0. \end{cases} \quad \ldots \quad (1)$$

As indicated in the classical literature [4], the fundamental theorem of statics (1) defines the so-called first form of equilibrium conditions for an arbitrary spatial system of forces [5]. At the same time, there are also the second and third forms of equilibrium [1] [3], which, however, are formulated exclusively for an arbitrary planar system of forces [6]. If a generalization of the third form of equilibrium to the case of an arbitrary spatial system of forces is of no practical or theoretical interest [7], then about a possible generalization of the second form of equilibrium conditions, which in the manual [3] is called *the three-moment theorem*, quite opposite associations arise. Indeed, the problems of geometric mechanics do not lose their relevance [8].

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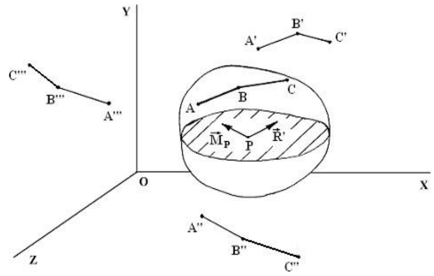


[Fig.1: Scheme of Static Equilibrium of a Rigid Body]

II. EXISTENCE THEOREM FOR A SYMMETRIC STATIC MOMENT TENSOR.

Consider the rigid body shown in Fig. 1 under the action of an arbitrary spatial system of forces. Without loss of generality of reasoning, we will assume that points A, B, and C form an angle $\angle A'B'C'$ on the XOY plane, and a segment $A''C'' = A''B'' \cup B''C''$ on the XOZ plane, where $A''B'' \neq 0$.

Existence theorem for a symmetric static moment tensor. A necessary and sufficient condition for the equilibrium of a rigid body under the action of an arbitrary spatial system of forces is the equality to zero of six components of the moments at three points of the rigid body that do not lie on one straight line, which form a symmetric moment tensor.

The necessity of the proof of the above theorem follows directly from the following vector relations:

$$\vec{M}_A = \vec{M}_P + \vec{AP} \times \vec{R}' = 0, \quad \vec{M}_B = \vec{M}_P + \vec{BP} \times \vec{R}' = 0, \quad \vec{M}_C = \vec{M}_P + \vec{CP} \times \vec{R}' = 0.$$

Sufficiency Let us have at points A, B and C the following groups of moments vanish concerning the axes Z, Y and X:

$$\begin{cases} M_A^z = M_B^z = M_C^z = 0; & \ldots \quad (2) \\ M_A^y = M_B^y = 0; & \ldots \quad (3) \\ M_A^x = 0. & \ldots \quad (4) \end{cases}$$

It is clear that, by conditions (2), the primary vector of the system of forces under consideration cannot simultaneously pass through the points A', B', C', and hence through their preimages A, B, C; however, conditions (2) do not at all entail the equality of the primary vector to zero, R'= 0, as is typical for a plan system of forces [1] [3].

It is easy to see that condition (2) implies the orthogonality of the primary vector R' to the XOY coordinate plane, i.e. in the case of collinearity of R to Z axis. Indeed,

$$\vec{M}_B = \vec{M}_A + \vec{BA} \times \vec{R}' \quad \text{and} \quad \vec{M}_C = \vec{M}_A + \vec{CA} \times \vec{R}',$$

or, after projecting onto the XOY plane,

$$M_B^z = M_A^z + \vec{B'A'} \times \vec{R}'_{xoy} \quad \text{and} \quad M_C^z = M_A^z + \vec{C'A'} \times \vec{R}'_{xoy},$$

whence, considering (2), as well as the fact that $B'C' \neq 0$, we will have

$$M_C^z - M_B^z = (C'\vec{A}' - B'\vec{A}') \times \vec{R}'_{xoy} = C'\vec{B}' \times \vec{R}'_{xoy} = 0 \quad \Rightarrow \quad \vec{R}'_{xoy} = 0,$$

Which is equivalent to the following result:

$$R'_x = R'_y = 0. \quad \cdots \quad (5)$$

We now turn to the consideration of equilibrium conditions (3) in the XOZ plane. As far as

$$M_B^y = M_A^y + B''\vec{A}'' \times \vec{R}'_{xoz} = 0,$$

Then, by the proposition $A'B' \neq 0$, we conclude that.

$$M_B^y - M_A^y = B''\vec{A}'' \times \vec{R}'_{xoz} = 0 \quad \Rightarrow \quad \vec{R}'_{xoz} = 0,$$

Which, considering (5), is equivalent to the result:

$$R'_z = 0. \quad \cdots \quad (6)$$

Turning to the vector of the primary moment, M_P, of an arbitrary spatial system of forces,

We are convinced that due to the pair theorems [1, 2], a fulfilment of conditions (2) and (3) is

Accompanied by the zeroing of the two components of the primary moment vector, M_P:

$$M_P^y = M_P^z = 0,$$

However, to nullify the third component, i.e. to fulfil the condition

$$M_P^x = 0,$$

It is necessary and sufficient to vanish the torque relative to the X axis at any of the fixed points A, B, or C, which is equivalent to the condition (4) that is already imposed.

By the above proof and based on previously imposed conditions (2) – (4), it seems possible to reformulate the fondamental theorem of statics (1) in the form of equality to zero of the symmetric static moment tensor:

$$[M] = \begin{bmatrix} M_A^x & M_A^y & M_A^z \\ M_A^y & M_B^y & M_B^z \\ M_A^z & M_B^z & M_C^z \end{bmatrix} = 0. \quad \cdots \quad (7)$$

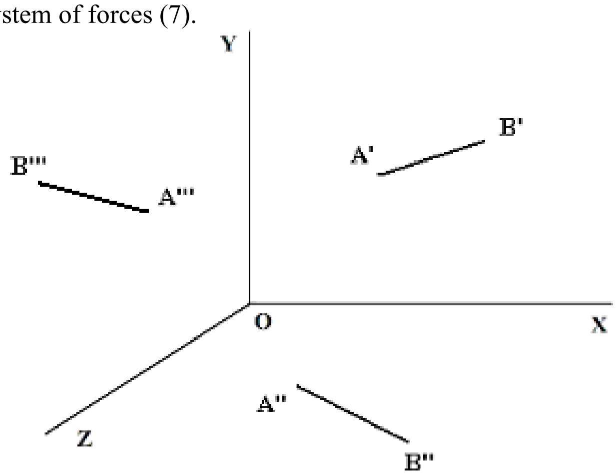
III. THE UNIQUENESS THEOREM FOR THE SYMMETRIC STATIC MOMENT TENSOR.

Again, we turn to the one shown in Fig. 1 to a solid body under the action of an arbitrary spatial system of forces. This time, it is required to prove the structural uniqueness of the static moment tensor (7).

The uniqueness theorem for the symmetric static moment tensor. Equality to zero of the static tensors of moments is the only structurally possible form of equilibrium in moments for an arbitrary spatial system of forces.

Proof. To prove the structural uniqueness of the static moment tensor, it is necessary to recall the central idea underlying the second form of the equilibrium condition about zeroing the primary moment at three points of a rigid body that do not lie on one straight line. Any assumption about a possible existence of some other kind of the second form of equilibrium of an arbitrary spatial system of forces through only two reference points of a rigid body does not stand up to criticism because the primary vector of the system of forces, as can be seen from Fig. 2 does not vanish. In the same way, it is impossible to describe such a form for which there could be more than three reference points on one of the coordinate planes.

Thus, for a strictly defined orientation of the rectangular Cartesian coordinate system axes as, for example, for the chosen right basic triple of vectors in Fig. 1 and 2, the structural uniqueness of the formation of a symmetric static equilibrium tensor is ensured, and hence the uniqueness of the form of equilibrium in moments for an arbitrary spatial system of forces (7).



[Fig.2: False Scheme of Static Equilibrium of a Rigid Body]

As a result of the above, it is curious to note that five components of the static moment. Tensor (7) is responsible for zeroing all components of the main vector R', while only three components of the same tensor are responsible for zeroing the primary moment M_P components.

IV. GENERALIZED ROTATION TENSOR

Based on the formulated static tensor of moments (7), designed to describe the static equilibrium of an arbitrary spatial system of forces, based on Newton's first law for steady motion and on the D'Alembert principle [6], it is not difficult to determine the *dynamic tensor of moments*:

$$[\tilde{M}] = \begin{bmatrix} \tilde{M}_A^x & \tilde{M}_A^y & \tilde{M}_A^z \\ \tilde{M}_A^y & \tilde{M}_B^y & \tilde{M}_B^z \\ \tilde{M}_A^z & \tilde{M}_B^z & \tilde{M}_C^z \end{bmatrix} = \begin{bmatrix} M_A^x - I_A^x\ddot{\varphi}_A^x & M_A^y - I_A^y\ddot{\varphi}_A^y & M_A^z - I_A^z\ddot{\varphi}_A^z \\ M_A^y - I_A^y\ddot{\varphi}_A^y & M_B^y - I_B^y\ddot{\varphi}_B^y & M_B^z - I_B^z\ddot{\varphi}_B^z \\ M_A^z - I_A^z\ddot{\varphi}_A^z & M_B^z - I_B^z\ddot{\varphi}_B^z & M_C^z - I_C^z\ddot{\varphi}_C^z \end{bmatrix}, \quad \cdots \quad (8)$$

where I и φ are the moment of inertia and the angle of rotation of the rigid body concerning.

The X, Y, Z coordinate axes pass through the points A, B, C.

Let us agree to call the static (7) and dynamic (8) moment tensors further the *generalized rotation tensor*, denoting the latter as [M], and its elements as m_{ij}. The vectors of moments at points A, B and C will be called the *primary moments* M_A, M_B, и M_C *of the generalized rotation about the centers of reduction A, B,*

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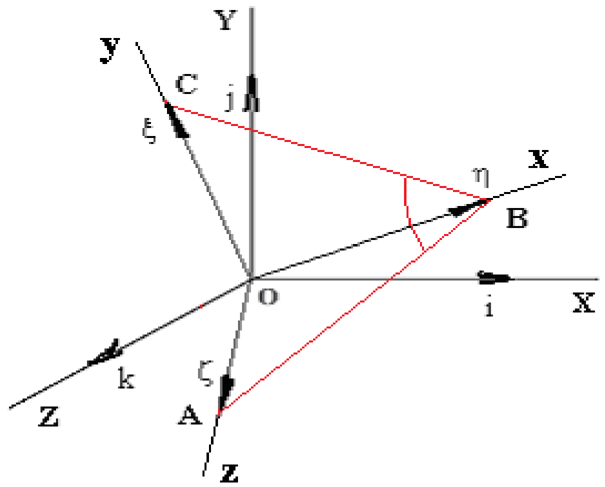
and C. When the centers of reduction of moments A, B, and C merge in a common center O, the generalized rotation of a rigid body degenerates into a traditional rotational motion around the center O.

V. INTERPRETATION OF THE TENSOR AS A RECTANGULAR ORTHONORMAL BASIS.

Given the foregoing, six definite values of the moments at three points of a rigid body that do not lie on a segment are necessary and sufficient to describe the state of static equilibrium of a rigid body. Consequently, these values can act as components of the symmetric moment tensor, expressed in the system of rectangular coordinate axes OXYZ of the right orientation of the orthonormal basis ijk in the form of a rectangular orthonormal basis $\eta\xi\zeta$ of the same orientation [7] (Fig. 3). And from this, in turn, we can conclude that the initially assumed $\angle ABC$ one can be associated with the angle formed between the ends of the orts of a rectangular orthonormal basis arbitrarily rotated in space relative to the immobile rectangular orthonormal basis of the same orientation (Fig. 3).



[Fig.3: Representation of the Moment Tensor as a Rectangular Orthonormal Basis]

Thus, the fundamental mathematical property of the tensor (second-order tensor) is confirmed as an invariant object uniquely defined at each point of a rigid body (inertia tensor, stress tensor), which in the system of Cartesian axes is characterized by nine numbers which are transformed when the coordinate axes are rotated according to the law [7]:

$$m'_{kl} = \alpha_{ki}\alpha_{lj}m_{ij}. \quad \dots \quad (9)$$

At the same time, it becomes clear that if the points B, C, and A in Fig. 3 do not denote the ends of the vectors η, ξ, ζ, respectively, but express a different non-degenerate arrangement, then conditions (2) – (4) will determine the same symmetrical rotation tensor (8) in its six independent components located on one side of the main diagonal.

VI. PRINCIPAL DIRECTIONS AND PRINCIPAL VALUES OF THE ROTATION TENSOR.

It is known that if a vector, when transformed using a tensor of second order as a linear operator, only changes its value by a factor of λ, but its direction does not change, i.e.

$$a_{ij}x_j = \lambda x_i,$$

Then this direction is called the *main direction of the tensor* [a], and the value λ is called the *principal value of the tensor* [a] [7].

The *principal directions* and *principal values of the rotation tensor* are found by solving a system of equations, which can be represented as follows:

$$(m_{ij} - \lambda\delta_{ij})x_j = 0.$$

Since the vector x indicating the main direction of the tensor cannot be zero, i.e.

x_1, x_2, x_3, cannot simultaneously vanish, then, therefore, the system of homogeneous linear equations should not have a zero solution. Therefore, the determinant from the coefficients of this system must be equal to zero [7]:

$$\left| m_{ij} - \lambda\delta_{ij} \right| = 0. \quad \dots \quad (10)$$

In expanded form, the *characteristic equation of the rotation tensor* [m] will look like this:

$$\begin{vmatrix} (m_{11}-\lambda) & m_{12} & m_{13} \\ m_{21} & (m_{22}-\lambda) & m_{23} \\ m_{31} & m_{32} & (m_{33}-\lambda) \end{vmatrix} = 0.$$

Expanding the determinant, we obtain a cubic equation with respect to λ [7], i.e.

$$\lambda^3 - I_1(m_{ij})\lambda^2 + I_2(m_{ij})\lambda - I_3(m_{ij}) = 0, \quad \dots \quad (11)$$

where $I_1, I_2,$ and I_3 are the *first, second, and third invariants of the rotation tensor* calculated from the following relations:

$$I_1(m_{ij}) = m_{11} + m_{22} + m_{33},$$

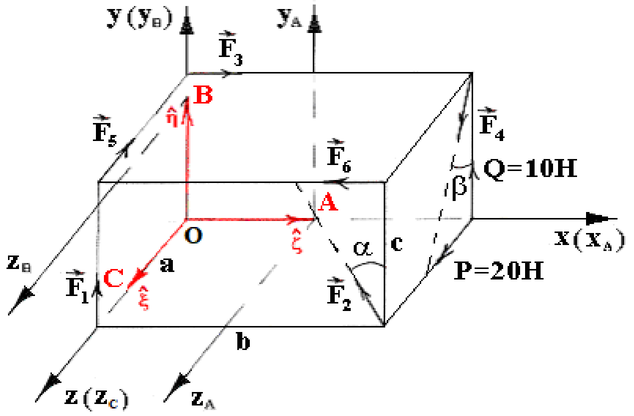
$$I_2(m_{ij}) = \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} + \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix} + \begin{vmatrix} m_{33} & m_{31} \\ m_{13} & m_{11} \end{vmatrix},$$

$$I_3(m_{ij}) = \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix}.$$

It is proved [7] that in the case of a symmetric second-order tensor [a], all three roots of the cubic equations are real. Consequently, the rotation tensor [m] has three real eigenvalues (11) $\lambda_1 \le \lambda_2 \le \lambda_3$ corresponding to the three main directions of the tensor along three mutually orthogonal principal axes, determined by the eigenvectors, which are found from the formula x_1, x_2, x_3 (10). The principal values of the rotation tensor determine the values of the three principal moments at points A, B, and C.

Since these are the central values of the symmetric rotation tensor $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$, $\lambda_3 = \mu_3$, measured along the axes Ox, Oy, Oz (Fig. 3), the equation of the *moment ellipsoid* takes the form [7]:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad \text{where} \quad a = \frac{1}{\sqrt{\mu_1}}, \quad b = \frac{1}{\sqrt{\mu_2}}, \quad c = \frac{1}{\sqrt{\mu_3}}. \qquad \ldots (12)$$



[Fig. 4. An Example of the Equilibrium of an Arbitrary Spatial System of Forces]

VII. CONTROL EXAMPLE OF NUMERICAL IMPLEMENTATION.

Consider the state of static equilibrium of an arbitrary spatial system of forces using the example shown in Fig. 4 rectangular parallelepipeds with overall dimensions: $a = 1.2$m, $b = 1.6$m, $c = 1.2$m. Angles α and β in fig. 4 are fixed by the values: $\alpha = 60^\circ$ and $\beta = 30^\circ$.

The fundamental theorem of statics (1) as applied to the system of forces shown in Fig. 4 is written in expanded form as follows:

$$\begin{cases} F_x = -F_2 \sin\alpha + F_3 - F_6 = 0, \\ F_y = F_1 + F_2 \cos\alpha - F_4 \cos\beta + Q = 0, \\ F_z = F_4 \sin\beta - F_5 + P = 0; \\ \hline M_{0x} = -F_1 a - F_2 a \cos\alpha + F_4 c \sin\beta - F_5 c = 0, \\ M_{oy} = -F_2 a \sin\alpha - F_4 b \sin\beta - F_6 a - Pb = 0, \\ M_{oz} = F_2 b \cos\alpha - F_3 c - F_4 b \cos\beta + F_6 c + Qb. \end{cases} \qquad \ldots (13)$$

By solving the system of linear algebraic equations (13) for forces $F_1, F_2, \ldots, F_6$, the required unknowns are found: $F_1 = -87.23$ N, $F_2 = 134.45$ N, $F_3 = -19.18$ N, $F_4 = -11.56$ N, $F_5 = 14.22$ N, $F_6 = -135.49$ N.

Then, the same problem is retested using the condition that the symmetric static moment tensor (7) is equal to zero as a criterion for static equilibrium, which is equivalent to zeroing its six independent components:

$$\begin{cases} M_{Ax} = -F_1 a - F_2 a \cos\alpha + F_4 c \sin\beta - F_5 c = 0, \\ M_{Ay} = -F_2 a \sin\alpha - F_4 (b-1) \sin\beta - F_5 - F_6 a - P(b-1) = 0, \\ M_{Az} = -F_1 + F_2 (b-1) \cos\alpha - F_3 c - F_4 (b-1) \cos\beta + F_6 c + Q(b-1) = 0; \\ \hline M_{By} = -F_2 a \sin\alpha - F_4 b \sin\beta - F_6 a - Pb = 0, \\ M_{Bz} = F_2 b \cos\alpha - F_2 \sin\alpha - F_3 (c-1) - F_4 b \cos\beta + F_6 (c-1) + Qb = 0; \\ \hline M_{Cz} = F_2 b \cos\alpha - F_3 c - F_4 b \cos\beta + F_6 c + Qb = 0. \end{cases} \qquad \ldots (14)$$

The secondary calculation of the control example based on the new equilibrium conditions (14) leads to an exact coincidence of the calculated values of the forces $F_1, F_2, \ldots, F_6$

With the results of the initial solution of the problem, carried out by the classical Poinsot theorem.

VIII. CONCLUSION

The presented generalization of the second form of the equilibrium conditions for an arbitrary plan system of forces to the case of an arbitrary spatial system of forces makes it possible to formulate the conditions for static equilibrium of a rigid body in the most concise form of equality to zero of the static tensor of moments.

The theoretical speculations presented in the article show that the second form of equilibrium conditions for an arbitrary spatial system of forces creates good prerequisites for describing the states of static and

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dynamic equilibrium with an appropriate choice of the directions of the cordinate axes using not six well-known equations of statics or motion, but only three. Then, the arbitrary motion of a rigid body can be identified with a generalized rotation of the considered rigid body. Meanwhile, the generalized rotation degenerates into ordinary rotational motion when the centers of reduction of moments A, B, and C merge into one common center of rotation O. The above generalization is of great interest in the field of studying problems of dynamics, for which it is difficult to overestimate the halving of the number of fundamental equations of motion.

Consequently, the scientific novelty of the newly introduced rotation tensor contains not only obvious theoretical, but also undeniable practical interest.

DECLARATION STATEMENT

I must verify the accuracy of the following information as the article's author.

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