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Estimating Poverty Measures using Truncated Distributions and its Applications

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***Abstract-** Poverty measures are used to measure poverty levels or degrees of poverty in a population. In this paper, we investigate the estimation of poverty measures using truncated log-normal, truncated gamma, and truncated epsilon-skew-normal distributions. For comparisons and illustrations, our results are applied to the real data sets collected in Egypt between 1995/1996 and 2008/2009.*

***Index Terms:** Poverty measures, Parametric estimation, Income distributions, Truncated distributions.*

I. INTRODUCTION

Poverty measure is defined by Duclos (2006) as the function of a given distribution F(x) of random variable X (income or expenditure) and a poverty line z, denoted by P(F, z) where x<z, and the poverty line z is a borderline between poor and non-poor people in a population. There are two common classes of poverty measures.

(a) Additively separable poverty measures. The class of additively separable poverty measures are defined by a given poverty line z and an individual deprivation function h(x, z) that is differentiable in both x and z for $x \in [0, z)$ and $z \in (0, +\infty)$. This class of poverty measures is decomposable across population subgroups. The indices of this class have the property of being expressible as a weighted sum (as separable function) and have the following form

$$P(F,z) = \int_0^{+\infty} I_{[0,z)}(x)h(x,z)dF(x), \qquad (1)$$

where the indicator function $I_{[0,z](x)}$ is defined by

$$I_{[0,z)}(x) = \begin{cases} 1 & x \in [0,z) \\ 0 & \text{elsewhere.} \end{cases}$$

The most popular index in this class is called Foster's index (Foster, 1984), in which h(x, z) is given by

Thus the Foster's index is defined as

$$h(x,z) = \left(1 - \frac{x}{z}\right)^{\alpha}, \quad \alpha \geq 0.$$

When α = 0, the Foster's index is called the Head-count ratio, denoted by H. When α = 1, it is called the Poverty gap ratio, denoted by P_1 (F, z). When α = 2, it is called the Severity of poverty, denoted by P_2 (F, z).

(b) Rank-based poverty measures. Rank-based Poverty measures use the position or the rank of each poor as an indicator of the relative deprivation function q(x, z) that is differentiable in both x and z for $x \in [0, z)$ and $z \in (0, \infty)$, and has the following form

$$P(F,z) = \int_0^{+\infty} I_{[0,z)}(x)q(x,z)dF(x). \qquad (3)$$

As an example, Sen (1976) proposed the following relative deprivation function

$$q(x,z) = 2\left(\frac{z-x}{z}\right)\left(1 - \frac{F(x)}{F(z)}\right),$$

and the index, called Sen's index later, is

$$S = 2\int_0^{+\infty} I_{[0,z)}(x)\left(\frac{z-x}{z}\right)\left(1 - \frac{F(x)}{F(z)}\right)dF(x). \qquad (4)$$

In addition, the Modified Sen's index (Shorrocks, 1995) is defined by

$$S_m = 2\int_0^{+\infty} I_{[0,z)}(x)\left(\frac{z-x}{z}\right)(1 - F(x))dF(x). \qquad (5)$$

In this paper, a brief review of income distribution models is given in Section 2. Truncated distributions and parameters estimation are discussed in Section 3. Truncated distributions to estimate poverty measures are investigated in Section 4. In Section 5, our main results are applied to the real survey data on household expenditures collected in Egypt (from Egyptian Central Agency of Statistics) between 1995/1996 and 2008/2009, as an illustration.

$$g_X(x) = \begin{cases} \frac{A^{-1}}{x} exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) & \text{if } 0 < x \leq T \\ 0 & \text{elsewhere,} \end{cases}$$

II. A BRIEF REVIEW OF INCOME DISTRIBUTION MODELS

As mentioned before, an estimation of income distributions is required to estimate poverty measures. Two approaches are used to estimate the income distribution.

(a) The first approach. This approach is a non-parametric approach which is easily understood and allows data to explain itself using histograms. The empirical distribution function is defined by

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$$\hat{F_n}(t) = \frac{1}{n}\sum_{i=1}^{n} I_{[0,t)}(X_i),$$

the most important non-parametric methods that is used to estimate the cumulative distribution function F(x) of random variable X (income or expenditure) and it is used to estimate the poverty measures which is defined in (1) or in (3), (see Kakwani (1993), Davidson (2000) and Berger (2003) for details). Also, the kernel method is a good tool in non-parametric approach to estimate the distribution of X and is defined by

$$\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^{n} \psi(\frac{x - x_i}{h}),$$

Where h is the window width and ψ is the kernel function which is usually, but not always, symmetric, see Zheng (2001) and Berger (2003).

(b)The second approach. This approach is a parametric approach which uses the parametric models of income distributions. These models assume that the income distribution follows a known functional form with unknown parameters. The pareto, log-normal, beta and gamma distributions are the most popular examples of the distribution of income, see Singh (1976), Salem (1974), Harrison (1981), and MacDonald (1984, 1995). In the literature examined, many distributions have been used to describe the distribution of income. In 1897, the distribution of income in the form of a probability density function was proposed by Vilfredo Pareto, see Harrison (1981). Pareto's function is accurately fitting high levels of income, but it is not good in describing the law end of the distribution. The log-normal distribution fits the lower levels of income better than the upper levels of income, see Singh (1976) and Harrison (1981). The log-normal density function takes the following form

$$f_X(x) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right) & \text{if } x > 0 \\ 0 & \text{elsewhere,} \end{cases}$$

Where $\mu \in \mathbb{R}$ and $\sigma > 0$are the mean and the variance of the normal distribution. Salem (1974) introduced the gamma distribution to fit United States income data. The gamma density function may take this form

$$f_X(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)}(x-c)^{\alpha-1}e^{-\beta(x-c)} & \text{if } x \geq c \\ 0 & \text{elsewhere,} \end{cases}$$

Where α and β are two positive parameters and c > 0 is a constant. In the next section truncated distributions and parameters estimation are discussed.

III. Truncated Distributions and Parameters Estimation

The truncated distributions have many applications in science. In this section truncated log-normal, truncated gamma, and truncated epsilon-skew-normal distributions are defined. Also, the parameters estimation of these distributions are discussed.

(i) Truncated log-normal distribution. The truncated log-normal distribution has a density function which may be written by

$$g_X(x) = \begin{cases} \frac{A^{-1}}{x} exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right) & \text{if } 0 < x \leq T \\ 0 & \text{elsewhere,} \end{cases}$$

Where

$$A = \int_0^T exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right) dx.$$

(ii) Truncated gamma distribution. The truncated gamma density function may take this form

$$g_X(x) = \begin{cases} A^{-1}(x-c)^{\alpha-1}e^{-\beta(x-c)} & \text{if } c \leq x \leq T \\ 0 & \text{elsewhere,} \end{cases}$$

Where c > 0 and

$$A = \int_c^T (x-c)^{\alpha-1}e^{-\beta(x-c)}dx.$$

(iii) Truncated epsilon-skew-normal distribution. The epsilon-skew-normal distribution proposed by Mudholkar (2000) has a probability density function written by

$$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2(1-\varepsilon)^2}\right) & \text{if } x < 0 \\ \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2(1+\varepsilon)^2}\right) & \text{if } x \geq 0 \end{cases}$$

Where θ=0 and σ=1. Then the truncated epsilon-skew-normal probability distribution can be written by

$$g_X(x) = \begin{cases} A^{-1} exp\left(-\frac{x^2}{2(1+\varepsilon)^2}\right) & \text{if } 0 \leq x \leq T \\ 0 & \text{elsewhere,} \end{cases}$$

Where

$$A = \int_0^T exp\left(-\frac{x^2}{2(1-\varepsilon)^2}\right) dx.$$

(a) Estimating the parameters of truncated distributions. In the literature reviewed, least-squares procedure and maximum likelihood estimation are two common methods that are used to estimate the parameters of statistical models and truncated distributions, see Chapman (1965), Salem (1974), Singh (1976), Slocomb (1977), Harrison (1981), Bandourian (2000), Mudholkar (2000), and Aban (2006). In this section, the parameters of truncated log-normal, truncated gamma, and truncated epsilon-skew-normal distributions are estimated.

(1) Method of least-squares procedure. Parameters of the truncated distributions are estimated by minimizing the mean squared error fit on a plot of the truncated distribution function see Chapman (1965) and Aban (2006). Let n is the observations be grouped by classes

$$(a_i - h_i, a_i + h_i),\ i = 1, \cdots, r,$$

Where

$$a_1 - h_1 = 0,\ a_r + h_r = T,\ a_i + h_i = a_{i+1} - h_{i+1},$$

Let $n_{\{i\}}$ is the number of observations falling in class i between $a_{\{i\}} - h_{\{i\}}$ and $a_{\{i\}} - h_{\{i\}}$ and

$$n = \sum_{i=1}^{r} n_i.$$

Define

$$p_i = \int_{a_i - h_i}^{a_i + h_i} g_X(x)dx \doteq g(a_i)(2h_i),$$

and

$$q_i = \frac{n_i}{n}.$$

Now, we can use the form

$$\ln p_i - \ln p_{i+1}, \quad i = 1, \cdots, r-1$$

to estimate the parameters of the truncated distributions with replacing $q_{\{i\}}$ instead of $p_{\{i\}}$, i=1,… ,r-1.The least-squares procedure is a common and a good method that is used to estimate the parameters of the truncated distributions when the income or expenditure data are grouped in intervals.

(2) Method of maximum likelihood estimation. For a fixed data set and a given statistical model, the maximum likelihood provides estimates for the model's parameters that maximize the likelihood function $L(x,\theta)$ defined by

$$L(x,\theta) = \prod_{i=1} g(x_i,\theta),$$

Where θ is the parameter and $g(x_i,\theta)$ is the probability densities function of the truncated distribution.

IV. TRUNCATED DISTRIBUTIONS TO ESTIMATE POVERTY MEASURES

As mentioned before, a poverty measure is defined as the function of a given distribution F(x) of random variable X (income or expenditure) and a poverty line z. We can conclude that the poverty measures depend on the low levels of income or expenditure; the left part of the distribution f(x), x < z. The truncated log-normal, truncated gamma, and truncated epsilon-skew-normal distributions then can be investigated to describe the distribution of the poor in a population and to estimate the poverty measures. The sum of squared errors (SSE) (Bandourian (2000)), the sum of absolute errors (SAE) (Chang (2008)), and Chi-square goodness of fit (Harrison (1981)) are three measures; that are used to evaluate the performance of the estimated truncated distributions and defined by

$$SSE = \sum_{i=1}^{r}\left(q_i - p_i(\hat{\theta})\right)^2, \quad SAE = \sum_{i=1}^{r}\left|q_i - p_i(\hat{\theta})\right|,$$

and

$$\chi^2 = \sum_{i=1}^{r}\left[\frac{\left(q_i - p_i(\hat{\theta})\right)^2}{p_i(\hat{\theta})}\right],$$

Where r is the number of income or expenditure's groups and $\hat{\theta}$ is the estimated parameters vector of the truncated distribution. The selected truncated distribution will be used to estimate the Foster poverty indices, Sen index, and modified Sen index given in (2), (4), and (5), respectively.

V. REAL DATA APPLICATIONS

The most important data sources to measure the poverty levels in Egypt are household surveys. Egyptian household surveys are available for the years 1995/1996, 1999/2000, 2004/2005 and 2008/2009. Each period is written as two years because each survey starts in mid-year of the first period and ends in mid-year of the second one. The title of the surveys for 1995/1996, 1999/2000, 2004/2005 and 2008/2009 was "The Research of Consumption and Expenditure in Egypt" and all households surveys are produced by the Egyptian Central Agency of Statistics (ECAS). In this paper, we will use household expenditure instead of income because many reported incomes in developing countries might be far less than real incomes. Also, in the developing countries, the income data is limited since many people don't report secondary sources of income. In our work, consumption expenditure data sets on urban areas and on rural areas are collected independently. Also for each consumption expenditure group the average expenditure $\overline{X}$, the average expenditure on food $\overline{X}_f$, the number of households NH, and the number of individuals NI are given.
The relative poverty line defined by a percentage of mean or median income or expenditure is used as the borderline between the poor and non-poor people see Oti (1990), Gustafsson (1996), Sahn (2000) and Zheng (2001). In this paper, The relative poverty line is estimated as

(a) 1/3 of the median of annual household expenditure, $z_{\{-\}}$, which serves as the minimum poverty line and

(b) 2/3 of the median of annual household expenditure, $z_{\{+\}}$, which serves as the maximum of poverty line. The relative poverty line is updated automatically over time when the expenditure changes. The poverty line for rural, urban, and total Egypt survey data between 1995/1996 and 2008/2009 are estimated and listed in Table 1. The parameters of the truncated gamma, truncated log-normal, and truncated epsilon-skew-normal distributions are estimated using the real data sets collected in Egypt from 1995/1996 to 2008/2009 and using the method of least-squares procedure. Based on the sum of squared errors (SSE), the sum of absolute errors (SAE), and chi-square goodness of fit measures, the truncated gamma distribution has the best performance of the data and will be used to describe the expenditure of the poor, see Figure 1, 2, 3, 4 and Figure 5. The estimated parameters of truncated gamma distribution, the sum of absolute errors (SAE), the sum of squared errors (SSE), and chi-square goodness of fit values are listed in Table 2.
Also, the estimated poverty measures using the truncated gamma distribution are listed in Table 3 and Table 4.

Figures and Tables

TABLE I

ESTIMATED RELATIVE POVERTY LINES USING RURAL, URBAN AND TOTAL SURVEY DATA BETWEEN 95/96 AND 08/09.

Area	1995/1996 Relative Poverty Line		1999/2000 Relative Poverty Line	
	z_-	z_+	z_-	z_+
Rural	1601.4	3202.9	2162.5	4325.0
Urban	2061.2	4122.3	2990.1	5980.2
Area	**2004/2005 Relative Poverty Line**		**2008/2009 Relative Poverty Line**	
	z_-	z_+	z_-	z_+
Rural	2664.6	5329.2	4371.3	8742.6
Urban	3577.7	7155.4	5325.0	10650.1
Total	2997.8	5995.6	4661.8	9323.6

TABLE II

APPROXIMATED PARAMETERS OF TRUNCATED GAMMA, THE SUM ABSOLUTE ERRORS, THE SUM SQUARE ERRORS, AND CHI-SQUARE VALUES USING RURAL, URBAN, AND TOTAL SURVEY DATA BETWEEN 95/96 AND 08/09.

	1995/1996				
area	$\hat{\beta}$	$\hat{\alpha}$	ASE	SSE	χ^2
Rural	0.00020	2.763	0.3648	0.0312	0.1031
Urban	0.00030	3.458	0.2540	0.0249	0.0711
	1999/2000				
area	$\hat{\beta}$	$\hat{\alpha}$	ASE	SSE	χ^2
Rural	0.00044	3.821	0.1460	0.0047	0.0289
Urban	0.00060	5.029	0.1586	0.0040	0.0303
	2004/2005				
area	$\hat{\beta}$	$\hat{\alpha}$	ASE	SSE	χ^2
Rural	0.00010	2.995	0.0762	0.0012	0.0066
Urban	0.00090	7.534	0.2395	0.0093	0.0751
Total	0.00013	3.118	0.0800	0.0014	0.0088
	2008/2009				
area	$\hat{\beta}$	$\hat{\alpha}$	ASE	SSE	χ^2
Rural	0.00005	2.620	0.5148	0.0590	0.2373
Urban	0.00002	2.846	0.2259	0.1075	0.0619
Total	0.00004	2.829	0.1802	0.0107	0.0454

TABLE III

ESTIMATED POVERTY MEASURES USING TRUNCATED GAMMA, RELATIVE POVERTY LINES, RURAL AND URBAN SURVEY DATA BETWEEN 95/96 AND 08/09.

Measures	1995/1996 Rural		1995/1996 Urban		1999/2000 Rural		1999/2000 Urban	
	z_-	z_+	z_-	z_+	z_-	z_+	z_-	z_+
$\hat{H}$	4.35%	23.47%	2.99%	20.74%	3.17%	21.94%	3.16%	25.91%
$\hat{P}_1$	1.20%	6.82%	0.72%	5.50%	0.74%	5.88%	0.64%	6.69%
$\hat{P}_2$	0.53%	3.07%	0.28%	2.29%	0.28%	2.43%	0.22%	3.93%
$\hat{S}$	1.81%	10.07%	1.16%	8.48%	1.20%	9.05%	1.11%	10.64%
$\hat{S}_m$	2.36%	12.19%	1.43%	10.03%	1.46%	10.62%	1.27%	12.11%
Measures	**2004/2005 Rural**		**2004/2005 Urban**		**2008/2009 Rural**		**2008/2009 Urban**	
	z_-	z_+	z_-	z_+	z_-	z_+	z_-	z_+
$\hat{H}$	2.50%	16.35%	1.92%	26.24%	4.58%	24.07%	2.59%	17.21%
$\hat{P}_1$	0.64%	4.39%	0.30%	6.02%	1.28%	7.06%	0.68%	4.57%
$\hat{P}_2$	0.27%	1.87%	0.08%	2.10%	0.58%	3.21%	0.28%	1.94%
$\hat{S}$	0.99%	6.71%	0.57%	10.40%	1.91%	10.37%	1.05%	7.01%
$\hat{S}_m$	1.27%	8.03%	0.60%	11.25%	2.51%	12.61%	1.34%	8.41%

TABLE IV

ESTIMATED POVERTY MEASURES USING TRUNCATED GAMMA, RELATIVE POVERTY LINES, AND TOTAL SURVEY DATA IN 04/05 AND 08/09.

Measures	2004/2005 Total		2008/2009 Total	
	z_-	z_+	z_-	z_+
$\hat{H}$	2.67%	17.39%	2.98%	18.47%
$\hat{P}_1$	0.68%	4.69%	0.79%	5.05%
$\hat{P}_2$	0.28%	1.99%	0.34%	2.19%
$\hat{S}$	1.06%	7.16%	1.20%	7.49%
$\hat{S}_m$	1.34%	8.55%	1.56%	9.15%

z	Relative frequencies *10^5	Truncated gamma *10^5
500	0.23	0.29
1500	3.35	4.11
2500	7.84	11.18
3500	13.64	18.6
4500	21.97	24.34
5500	26.55	27.61
6500	26.43	28.49

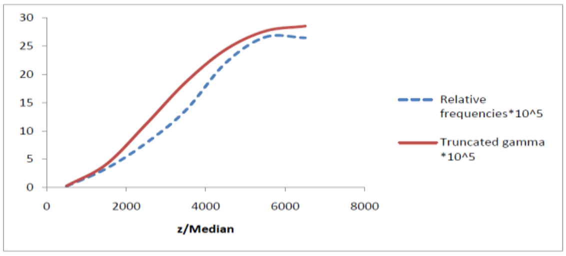

Fig. 1. Truncated gamma distribution - Rural 99/00.

This figure displays the truncated gamma distribution for rural data collected in 1999/2000. It shows that the truncated gamma distribution has the best performance of the data comparing with the relative frequency distribution.

z	Relative frequencies*10^5	Truncated gamma *10^5
1000	0.455	0.962
2500	4.337	5.15
3500	8.603	9.12
4500	14.432	13.62
5500	20.455	18.39
6500	24.467	23.22
7500	25.889	27.95

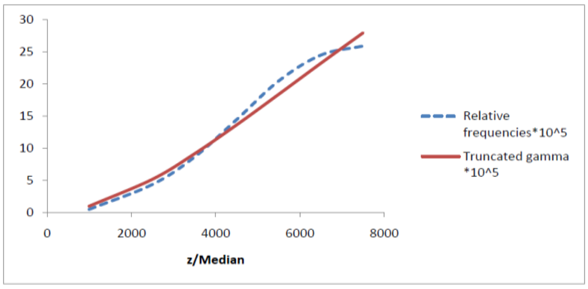

Fig. 2. Truncated gamma distribution - Rural 04/05.

This figure displays the truncated gamma distribution for rural data collected in 2004/2005. It shows that the truncated gamma distribution has the best performance of the data comparing with the relative frequency distribution.

z	Relative frequencies*10^5	Truncated gamma *10^5
500	0.025	0.022
1500	1.35	1.03
2500	3.57	4.43
3500	6.44	9.42
4500	10.84	14.24
5500	14.72	17.54
6500	16.28	18.87
7500	16.75	18.43
9000	7.51	15.62

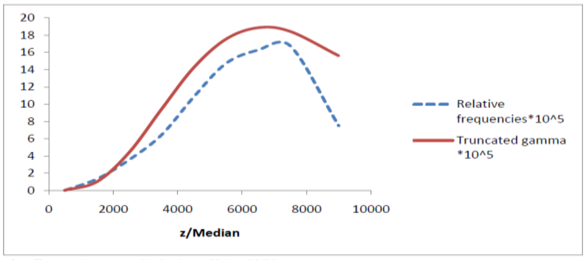

Fig. 3. Truncated gamma distribution - Urban 99/00.

This figure displays the truncated gamma distribution for urban data collected in 1999/2000. It shows that the truncated gamma distribution has the best performance of the data comparing with the relative frequency distribution.

z	Relative frequencies*10^5	Truncated gamma *10^5
1000	0.155	0.012
2500	1.72	1.19
3500	3.85	4.38
4500	6	9.19
5500	9.67	13.87
6500	12.4	16.79
7500	14.17	17.39
8500	15.94	16.02
9500	15.23	13.47
10750	9.05	9.81

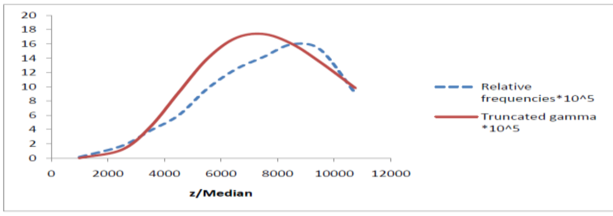

Fig. 4. Truncated gamma distribution - Urban 04/05.

This figure displays the truncated gamma distribution for urban data collected in 2004/2005. It shows that the truncated gamma distribution has the best performance of the data comparing with the relative frequency distribution.

z	Relative frequencies*10^5	Truncated gamma *10^5
1000	0.32	0.69
2500	3.2	3.92
3500	6.58	7.03
4500	10.79	10.51
5500	15.92	14.11
6500	19.51	17.65
7500	21.22	20.98
8500	21.47	24.01

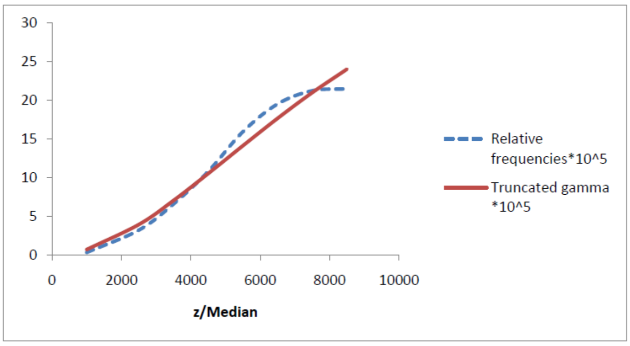

Fig. 5. Truncated gamma distribution - Total 04/05

This figure displays the truncated gamma distribution for total data collected in 2004/2005. It shows that the truncated gamma distribution has the best performance of the data comparing with the relative frequency distribution.

VI. CONCLUSION

The parameters of the truncated gamma, truncated log-normal, and truncated epsilon-skew-normal distributions are estimated using the real data sets collected in Egypt from 1995/1996 to 2008/2009 and using the method of least-squares procedure. Based on the sum of squared errors (SSE), the sum of absolute errors (SAE), and chi-square goodness of fit measures, the truncated gamma distribution has the best performance of the data and will be used to describe the expenditure of the poor. Also, the estimated poverty measures using the truncated gamma distribution are calculated.

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